MMP for algebraically integrable foliations

joint w/ F. Ambro, P. Cascini, V.V. Shokurov.

<u>Def</u> X normal variety

a foliation $\mathcal{F}$ is a coherent subsheaf

$$T_{\mathcal{F}} \subseteq T_X$$

i) $T_X / T_{\mathcal{F}}$ torsion free

ii) $[T_{\mathcal{F}}, T_{\mathcal{F}}] \subseteq T_{\mathcal{F}}$

$\operatorname{rank} \mathcal{F} = \operatorname{rank} T_{\mathcal{F}}$

Remark

i) bir'l notion.

A foliation is uniquely determined by its restriction to a Zar open subset.

$X, \mathcal{F}$

$X \overset{\phi}{\dashrightarrow} X'$ bir'l

$\exists$ a strict transform fol

$\phi^{-1}_{*} \mathcal{F}$

ii) if $f : X \dashrightarrow Y$ dominant ratl map then the fibres of f def.in a foliation, precisely,

$$\begin{tikzcd} U \subseteq X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad T_{\mathcal{F}}|_U = \ker df = T_{U/Y}.$$

iii) Mantra: morphism don't transform, but foliations do.

iv) We say a foliation is <u>alg. integrable</u> if it arises from a dominant rat'l map $X \dashrightarrow Y$

very few foliations are alg. integrable!

Usually leaves are transcendental

Frobenius theorem

$X, \mathcal{F}$ foliated variety

For general $x \in X$ $\exists$ euclidean nbhd $U \ni x$ and a holomorphic submersion

$U \to V$ s.t. $T_{\mathcal{F}}|_U = T_{U/V}$

Mori theory seems to work for foliations ...

Thm (Miyaoka, Bogomolov-McQuillan, Campana-Păun)
$\det(T_{\mathcal{F}})^*$ is not pseudo-effective, e.g. $\exists$ a movable curve C s.t.
$\det(T_{\mathcal{F}})^* \cdot C < 0$,

then $\exists$ a subfoliation
$\mathcal{E} \subseteq \mathcal{F}$ s.t.

i) $\mathcal{E}$ alg. integrable

ii) closure of a general leaf is rat'lly connected

Def $K_{\mathcal{F}}$ weil divisor
s.t. $\mathcal{O}(K_{\mathcal{F}}) = (\det T_{\mathcal{F}})^*$

cf. K_X not psef $\Rightarrow$
X uniruled

$K_{\mathcal{F}}$ not psef $\Rightarrow$
$\mathcal{F}$ uniruled.

Other results ...
on 2,3-folds MMP
for foliations works
(no assumption of alg.
integrable foliations)

Examples

- $X \to Z$ smooth morphism and $\mathcal{F}$ is the associated foliation

$$K_{\mathcal{F}} = K_{X/Z}$$

- $X \to Z$ equidimensional

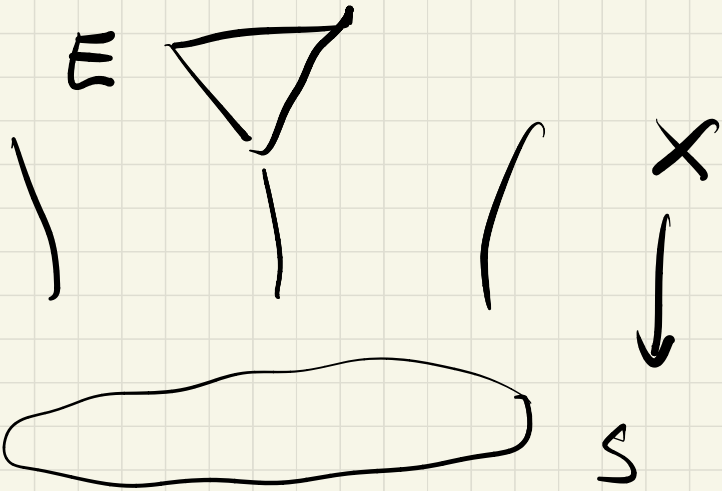
$$K_{\mathcal{F}} = K_{X/Z} - \sum_{F \text{ vertical}} (l_F - 1) F$$

mult l_F

-

$$
\begin{tikzcd}
Y = \mathbb{P}^1 \times S \arrow[r] & S \\
X \arrow[u, "b"'] \arrow[ur]
\end{tikzcd}
$$

blow up a pt



$$K_{X/S} = b^* K_{Y/S} + 2E$$

$$K_{y_x} = b^* K_{Y/S} + E$$

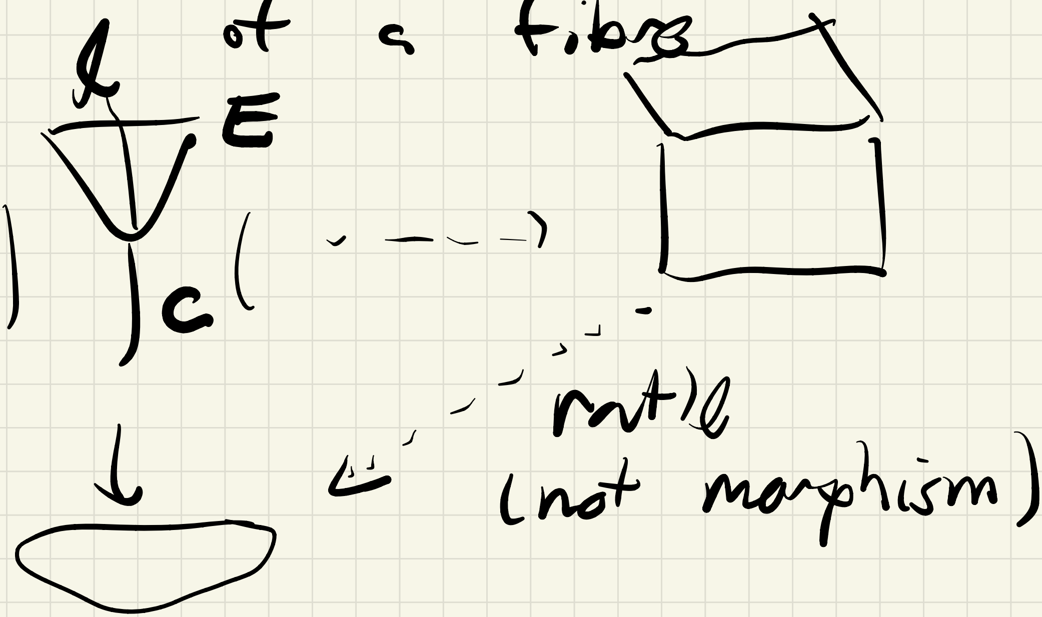
up shot ---

relative MMP $\neq$ foliated MMP

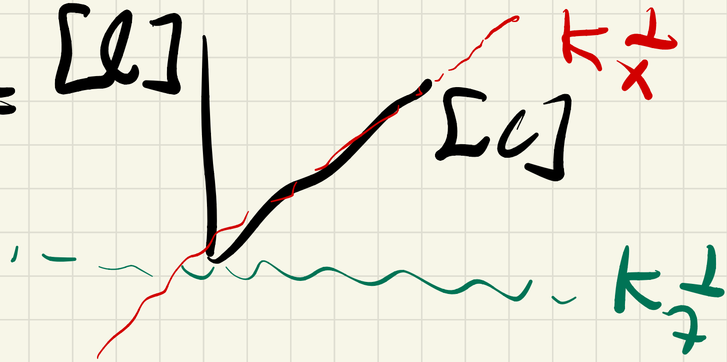
relative $K_{X/S}$-MMP contracts E to a point

$K_{\mathcal{F}}$-MMP

- E to a point
- flip strict transform of a fibre



$\overline{NE}(X/S) =$



by taking ramified covers

$\exists$ examples when

$K_{X/S}$ nef, but K_Y is not.

Q: When are the relative $K_{X/S}$-MMP and K_Y-MMP the same?

<u>Def</u> Property $(*)$

$(X, B) \xrightarrow{f} Z$ has $(*)$

1) Z to be smooth

2) (X, B) log canonical

3) $\exists$ $\Sigma \subseteq Z$ snc s.t.

$f^{-1}(\Sigma) = B^{v}$

4) for any $D \subseteq Z$ s.t.

$D + \Sigma$ snc

$(X, B + f^{*}D)$ lc.

<u>Remark</u> f equidimensional

+ reduced fibres

equivalent to local stability

Thm (ACSS) if

$f:(X,B) \to Z$ satisfies $(*)$

Then

$$K_Y + B^h = (K_X + B) - f^*(K_Z + B_Z)$$

discriminant (depends on sing of fibres)

and moreover

$K_Y + B^h$ -MMP

is a relative

$K_X + B$ -MMP.

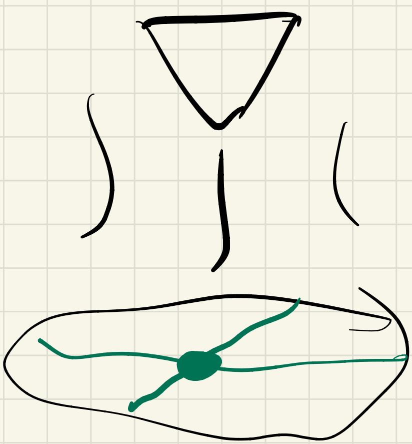
Remark relative MMP

gives relative netness

But ... $K_{\mathcal{F}} + B^h$ is globally nef (on all of X)

<u>Pf</u> replaced Hodge theory by foliated bend + break. $\blacksquare$

<u>Ex</u> not $(*)$



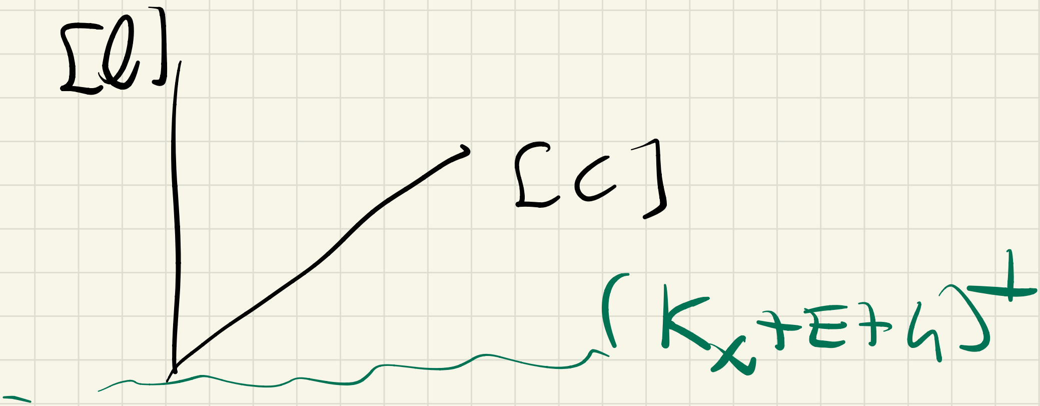
D

$$(X, f^*D) = (X, 2E + G)$$

nct lc

$$(X, E + G) \longrightarrow Z$$

$$\Sigma = D$$



<u>Cor</u> Existence of $(*)$ modifications
(Think alt modification)

<u>Cor</u> Cone theorem for log canonical foliated pair $(\mathcal{F}, B)$ when $\mathcal{F}$ alg. integrable.

A pair $(\mathcal{Y}, B)$ is log canonical if

$K_{\mathcal{Y}} + B$ $\mathbb{Q}$-Cartier

and for all bir'l morphism

$\pi : X' \longrightarrow X$

$$K_{\mathcal{Y}'} + \pi_*^{-1} B = \pi^*(K_{\mathcal{Y}} + B) + \sum a_E(\mathcal{Y}, B) E$$

$$a_E(\mathcal{Y}, B) \geq -\varepsilon(E)$$

$$\varepsilon(E) = \begin{cases} 1 & \text{if } E \text{ not inv.} \\ 0 & \text{if } E \text{ is inv.} \end{cases}$$

E invariant means

E is a union of leaves

($\mathcal{F}$ alg. integrable

E is vertical for fibration)

<u>Remark</u> $(\mathcal{F}, B)$ log canonical

A ample divisor then in general for any

$D \sim_{\mathbb{Q}} A$

$(\mathcal{F}, B+D)$ <u>not</u> log canonical.

- Bertini fails for toroidal morphisms.

- Base point free fails for foliations.
  (even for alg. int)

Thm (Cascini - S.) X klt $\mathbb{Q}$-fact.
$(\mathcal{F}, B)$ lc alg. integrable foliated pair, assume termination of log flips in dim $\leq$ rank $\mathcal{F}$, then $\exists$ $K_{\mathcal{F}}+B$ MMP

$$\phi : X \dashrightarrow X'$$, i.e.

a sequence of flips/div. contractions s.t.

either

i) $K_{\mathcal{F}'}+B'$ is nef

ii) $X' \xrightarrow{f} Z$

$-(K_{\mathcal{F}'}+B')$ f-ample.

<u>Pf</u> Hardest part is proving contraction thm.

$$
\begin{tikzcd}
(\bar{X}, \bar{\mathcal{F}}, \bar{B}) \arrow[r, dashed, "K_{\bar{\mathcal{F}}}+\bar{B}\text{-MMP}"'] \arrow[d, "(*)"'] & (\bar{X}', \bar{\mathcal{F}}', \bar{B}') \\
(X, \mathcal{F}, B) &
\end{tikzcd}
$$

use termination here.

run a smart

$K_{\bar{X}'} +$ [illegible] MMP

to contract all the divisors extracted by

(*) modification

$$\bar{X}' \dashrightarrow X^+$$

X^+ is the flip/divisorial contraction.

H_R is a supporting hyperplane to extremal ray

$mH_R - K_X$ one is ample

$mH_R^+ - K_{X^+}$

$\Rightarrow$ H_R semi-ample.

$$\begin{array}{c} C \times C \supseteq \Delta \qquad g(C) \geq 2 \\ \downarrow \\ C \end{array}$$

$K_{C \times C / C} + \Delta$ big, nef, not semi-ample

$$(K_{C \times C / C} + \Delta) \cdot \Delta = 0$$

$\Longrightarrow$ bpf false for foliations

$\underline{Q}$ $(Y, B+A)$ lc

$B \geq 0$ A ample

$K_Y + B + A$ big + nef

$\Rightarrow$ semi-ample?

3 folds yes.

pset ±Kx w,A K7